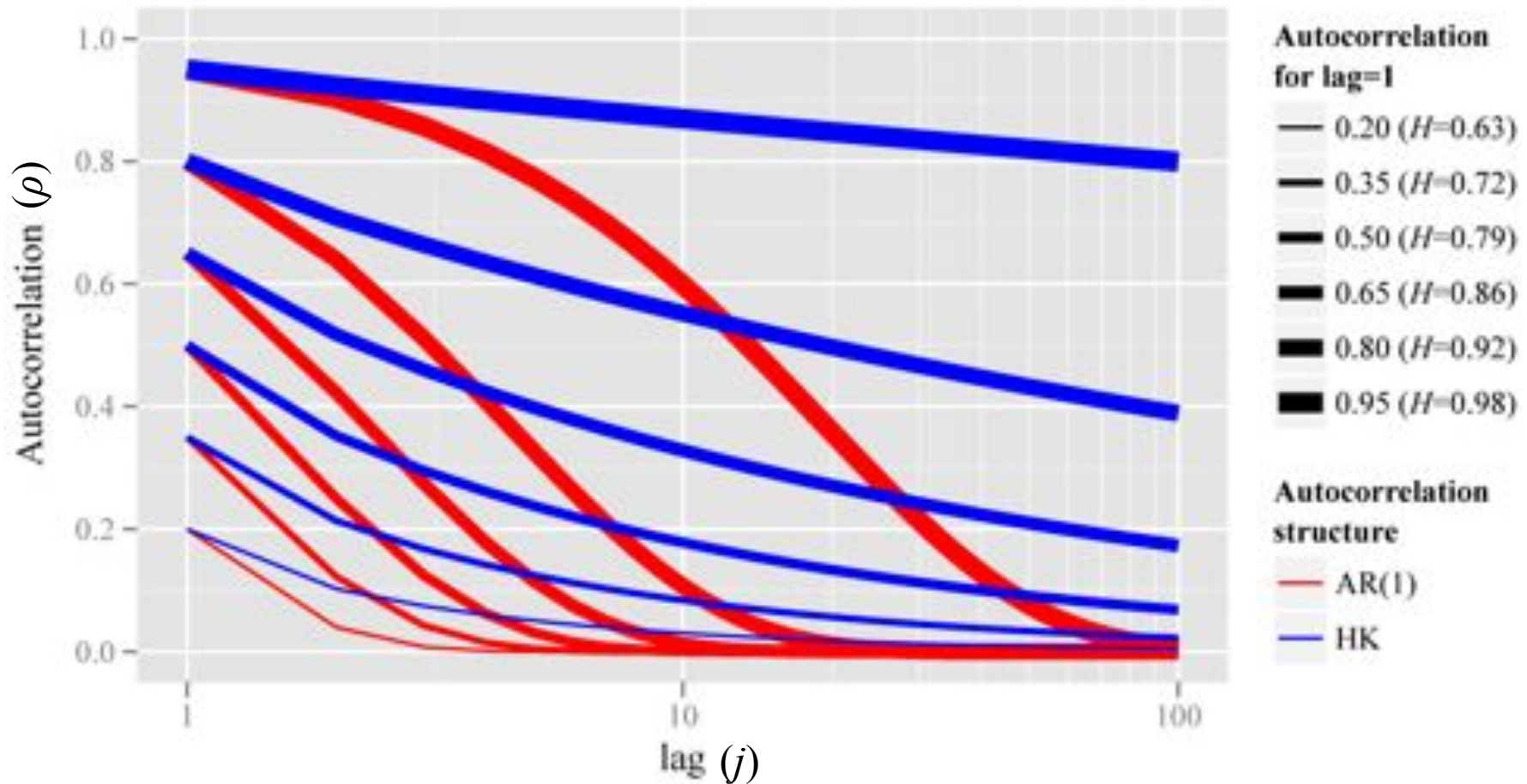

A quick gap-filling of missing hydrometeorological data

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Exponential ACS $\rho_j = \rho^{|j|}$

:

Power-type ACS : $\rho_j = \frac{1}{2} \left[(j+1)^{2H} + (j-1)^{2H} \right] - j^{2H}$

- Given that $2 \times N$ **observations** are available, we want to estimate a **missing value** y :

$$x_{-N}, \dots, x_{-1}, \boxed{y}, x_1, \dots, x_N$$

- A (linear) **estimate of y** can be expressed as:

$$\underline{y} = w_{-N}x_{-N} + \dots + w_Nx_N + \underline{e}$$

where

- x_i : the observed values
- w_i : weighting factors
- \underline{e} : estimation error

- The **Mean Squared Error** of the estimation is then defined as:

$$\text{MSE} := \text{E} [e^2] = \text{E} \left[(y - \underline{y})^2 \right]$$

- We examine the following **estimate for y** :

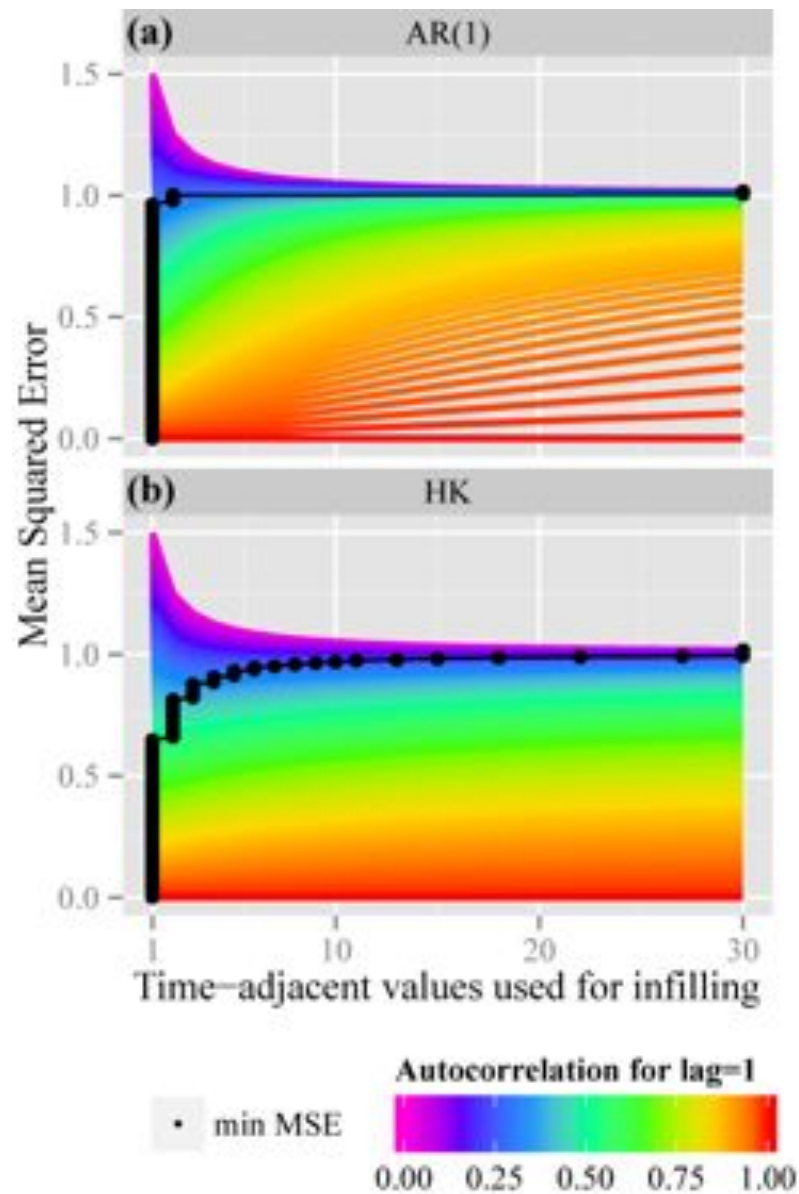
$$\underline{y} = \frac{\sum_{i=1}^n x_{-i} + \sum_{i=1}^n x_i}{2n}$$

$$x_{-N}, \dots, x_{-n}, \dots, x_{-1}, \boxed{y}, x_1, \dots, x_n, \dots, x_N$$

- Assuming that the underlying process is (weakly) **stationary**, the **MSE** of the estimation is given by:

$$\begin{aligned} \text{MSE} &:= E[e^2] = E\left[(y - \underline{y})^2\right] = E\left[\left(y - \frac{\sum_{i=1}^n x_{-i} + \sum_{i=1}^n x_i}{2n}\right)^2\right] \\ &= \frac{1}{2} \left(\frac{\sigma}{n}\right)^2 \left[(2n+1) \left(n - 2 \sum_{i=1}^n \rho_i\right) + \sum_{i=1}^{2n} (2n+1-i) \rho_i \right] \end{aligned}$$

- Which is the **optimal** (i.e., minMSE) number of **neighbouring values (n)** that should be used?



$$\underline{y} = \frac{\sum_{i=1}^n x_{-i} + \sum_{i=1}^n x_i}{2n}$$

MSE

- **AR(1)**

For a wide range of **lag-1 autocorrelations**, the strictly local average (i.e., $n=1$) provides the **minMSE**.

- **HK**

As the **lag-1 autocorrelation increases**, the time-adjacent values (n) required for a **minMSE** gradually decrease.

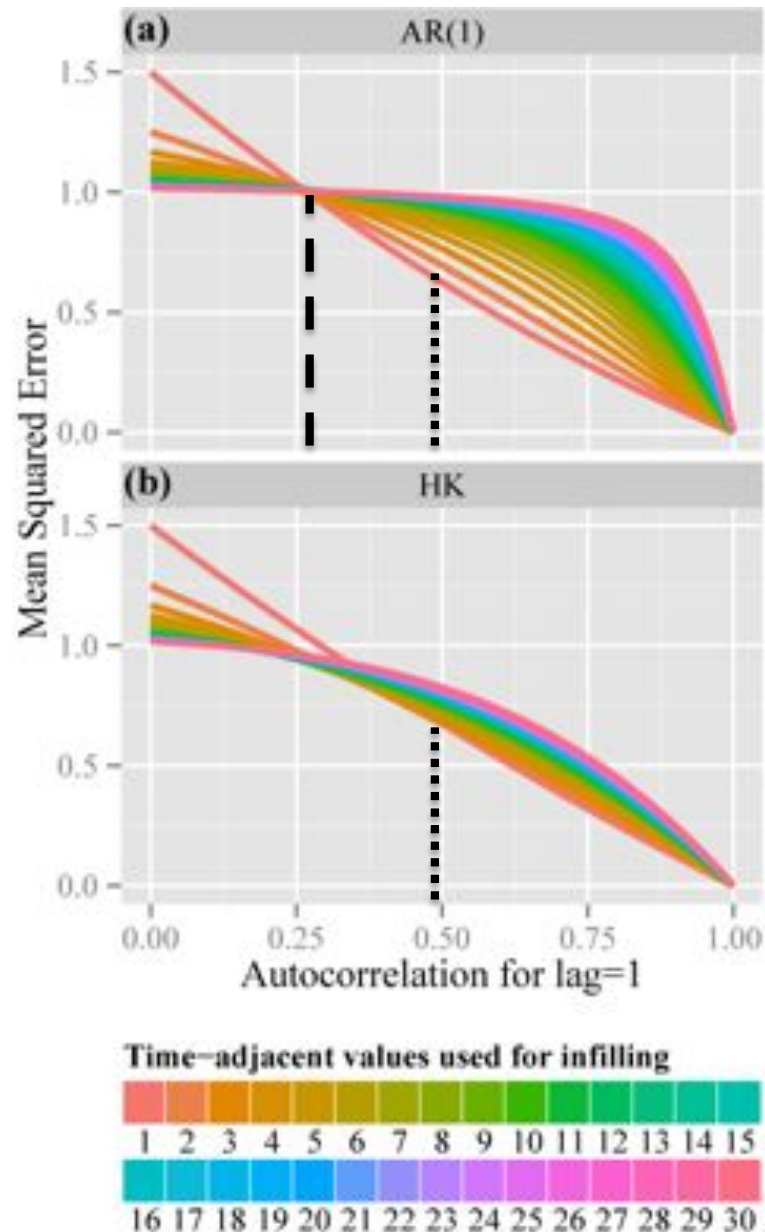
See also

Dialynas et al. (2010)

<http://itia.ntua.gr/en/docinfo/981/>

Pappas (2010)

<http://itia.ntua.gr/en/docinfo/1065/>



$$\underline{y} = \frac{\sum_{i=1}^n x_{-i} + \sum_{i=1}^n x_i}{2n}$$

MSE

Markovian property:

“The future does not depend on the past when the present is known” [Papoulis, 1965,

Optimal Local Average			
Short-term persistence -AR(1)-		Long-term persistence -HK-	
$\rho \leq 0.25$	$n=n_{\max}$	$\rho \leq 0.3$	$n=n_{\max}$
$0.25 < \rho \leq 0.28$	$n=2$	$0.30 < \rho \leq 0.32$	$n=4$
		$0.32 < \rho \leq 0.38$	$n=3$
$\rho > 0.28$	$n=1$	$0.38 < \rho \leq 0.51$	$n=2$
		$\rho > 0.51$	$n=1$

ρ : lag-one autocorrelation coefficient

n : time-adjacent values used for the infilling

n_{\max} : all the available observed values, i.e., total/sample average

For both ACS (exponential or power-type) when $\rho > 0.51$ the strictly local average ($n=1$) provides the minMSE.

- **Generalization** of the OLA methodology, so that **information** from both **local** and **global average** will be used according to the **lag-1 autocorrelation**.

- We examine the following **estimate for y** :

$$\bar{y} = \lambda \underbrace{\frac{\sum_{i=-N}^N x_i}{2N}}_{\text{Total (sample) average}} + (1 - \lambda) \underbrace{\frac{x_{-1} + x_1}{2}}_{\text{Local (strictly) average}}$$

where λ is the **weighting factor** for the **total (sample) average** and the **local (strictly) average**.

- Parameter λ reflects the strength of the temporal autocorrelation:

low values → **high correlation**

high values → **low correlation**

See also:

Pappas (2010)

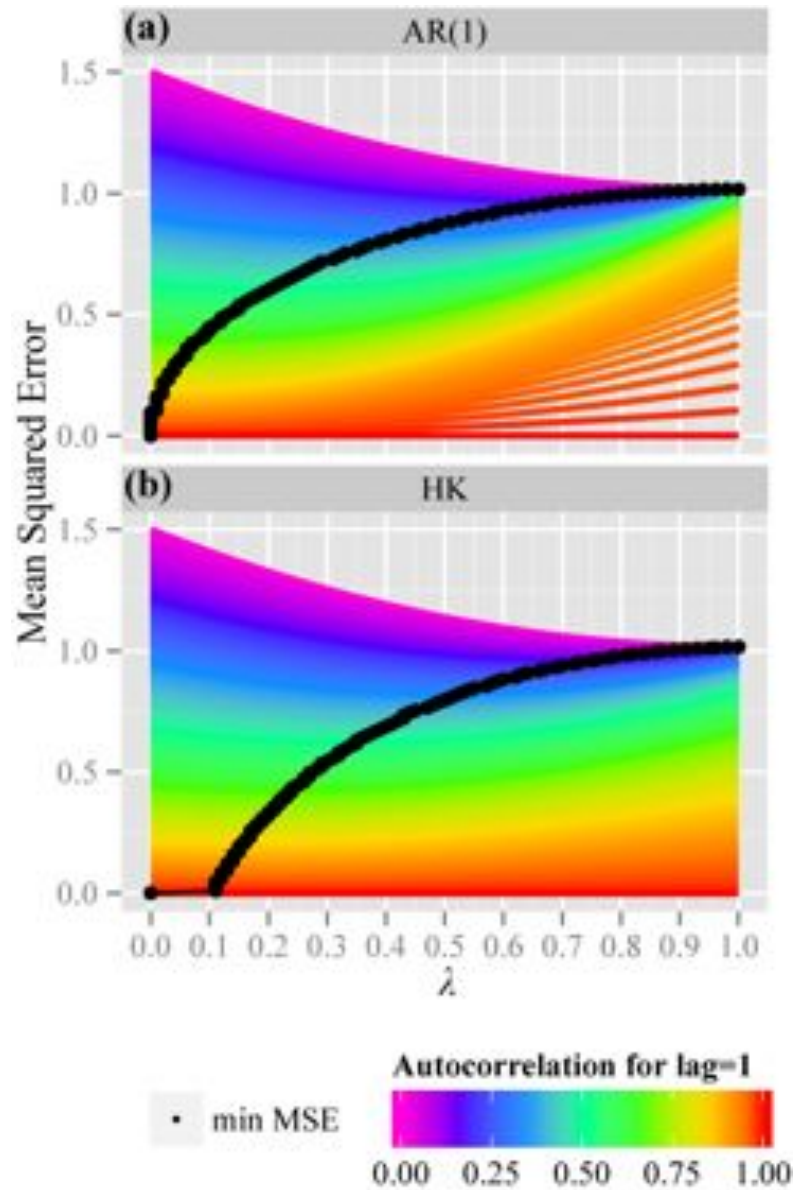
<http://itia.ntua.gr/en/docinfo/1065/>

- Assuming that the underlying process is (weakly) **stationary**, the **MSE** of the estimation is then defined as:

$$\text{MSE} := \text{E} [e^2] = \text{E} \left[(y - \underline{y})^2 \right] = \text{E} \left[\left(y - \left(\lambda \frac{\sum_{i=-N}^N x_i}{2N} + (1-\lambda) \frac{x_{-1} + x_1}{2} \right) \right)^2 \right]$$

- After some **algebraic** manipulations:

$$\begin{aligned} \text{MSE} = & \frac{1}{2} \sigma^2 (3 - 4\rho_1 + \rho_2) - 2\lambda\sigma^2 \left[\frac{1}{N} \sum_{i=1}^N \rho_i - \frac{1}{2N} \left(\sum_{i=1}^{N-1} \rho_i - \sum_{i=2}^{N+1} \rho_{i+1} \right) - \rho_1 + \frac{\rho_2}{2} + 0.5 \right] \\ & + \lambda^2 \sigma^2 \left[\frac{1}{2N^2} \left(2 \sum_{i=1}^{N-1} (N-i) \rho_i + \sum_{i=2}^{N+1} (i-1) \rho_i + \sum_{i=N+2}^{2N} (2N+1-i) \rho_i + N \right) \right. \\ & \left. + \frac{\rho_2}{2} + \frac{1}{2} - \frac{1}{N} \left(\sum_{i=1}^{N-1} \rho_i + \sum_{i=2}^{N+1} \rho_{i+1} \right) \right] \end{aligned}$$



$$\underline{y} = \lambda \frac{\sum_{i=-N}^N x_i}{2N} + (1 - \lambda) \frac{x_{-1} + x_1}{2}$$

• **AR(1) & HK**

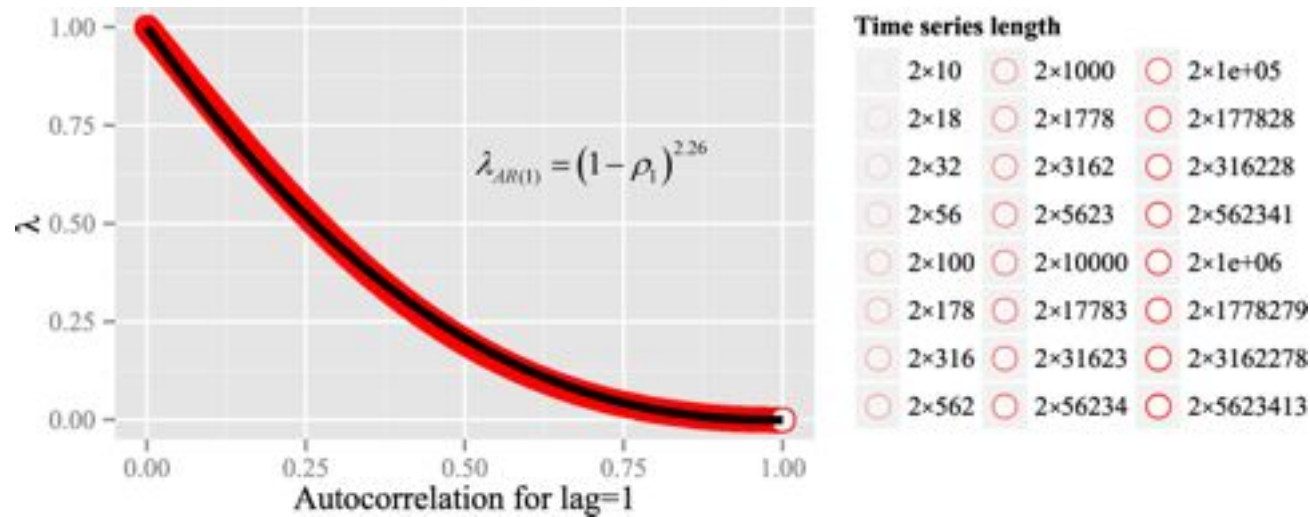
As **lag-1 autocorrelation** increases, the contribution of the local average increases (i.e., lower values of λ).

• **HK**

It *takes time* for the **HK process** to reveal its **properties**.

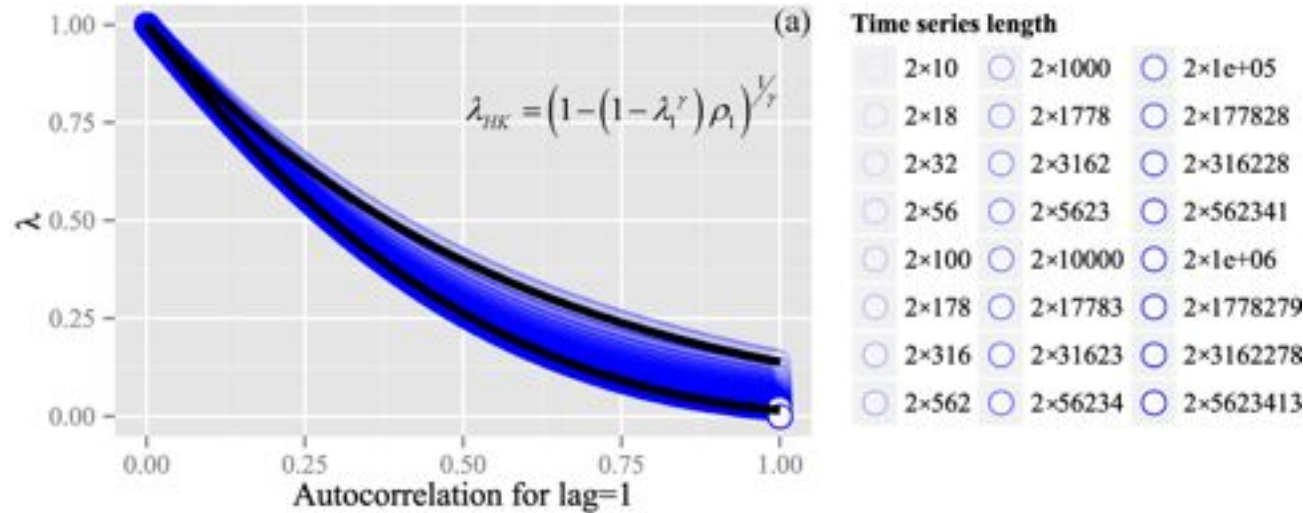
• The influence of **sample size (N)** :

$$\begin{aligned} \text{MSE} = & \frac{1}{2} \sigma^2 (3 - 4\rho_1 + \rho_2) - 2\lambda \sigma^2 \left[\frac{1}{N} \sum_{i=1}^N \rho_i - \frac{1}{2N} \left(\sum_{i=1}^{N-1} \rho_i - \sum_{i=2}^{N+1} \rho_{i+1} \right) - \rho_1 + \frac{\rho_2}{2} + 0.5 \right] \\ & + \lambda^2 \sigma^2 \left[\frac{1}{2N^2} \left(2 \sum_{i=1}^{N-1} (N-i) \rho_i + \sum_{i=2}^{N+1} (i-1) \rho_i + \sum_{i=N+2}^{2N} (2N+1-i) \rho_i + N \right) \right. \\ & \left. + \frac{\rho_2}{2} + \frac{1}{2} \left(\frac{1}{N} \sum_{i=1}^{N-1} \rho_i + \sum_{i=2}^{N+1} \rho_{i+1} \right) \right] \end{aligned}$$

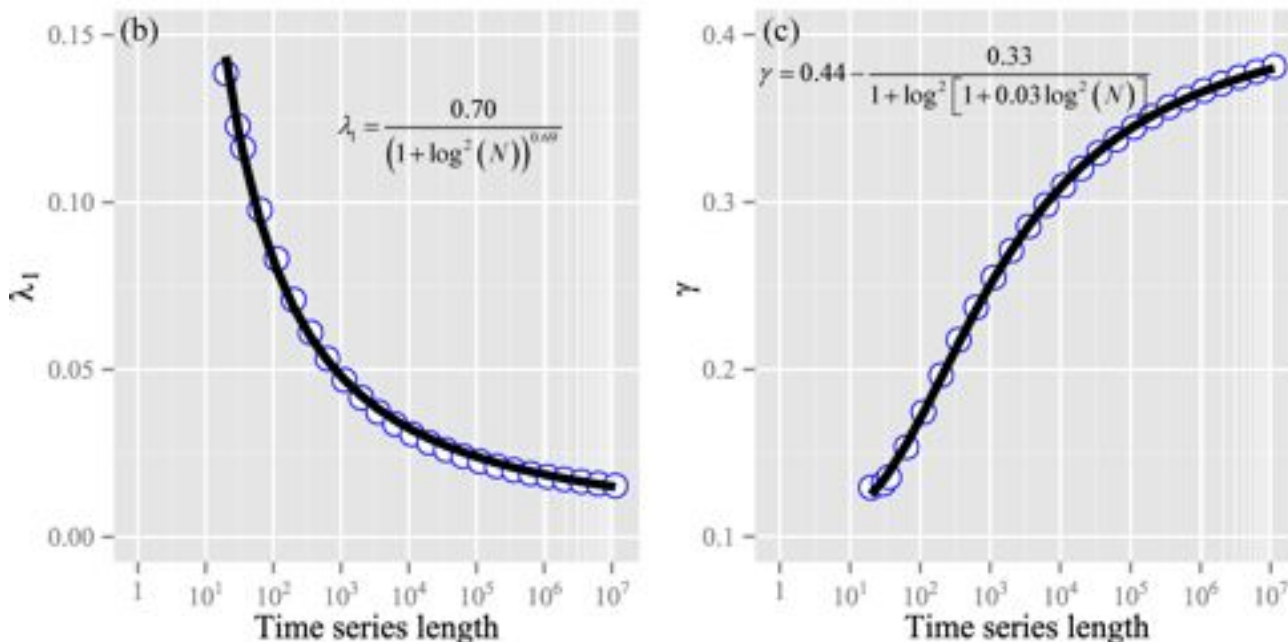


$$\underline{y} = \lambda_{AR(1)} \frac{\sum_{i=-N}^N x_i}{2N} + (1 - \lambda_{AR(1)}) \frac{x_{-1} + x_1}{2}$$

- For the case of **exponential ACS**, the λ - ρ_1 relationship does **not vary** significantly with time series length N .

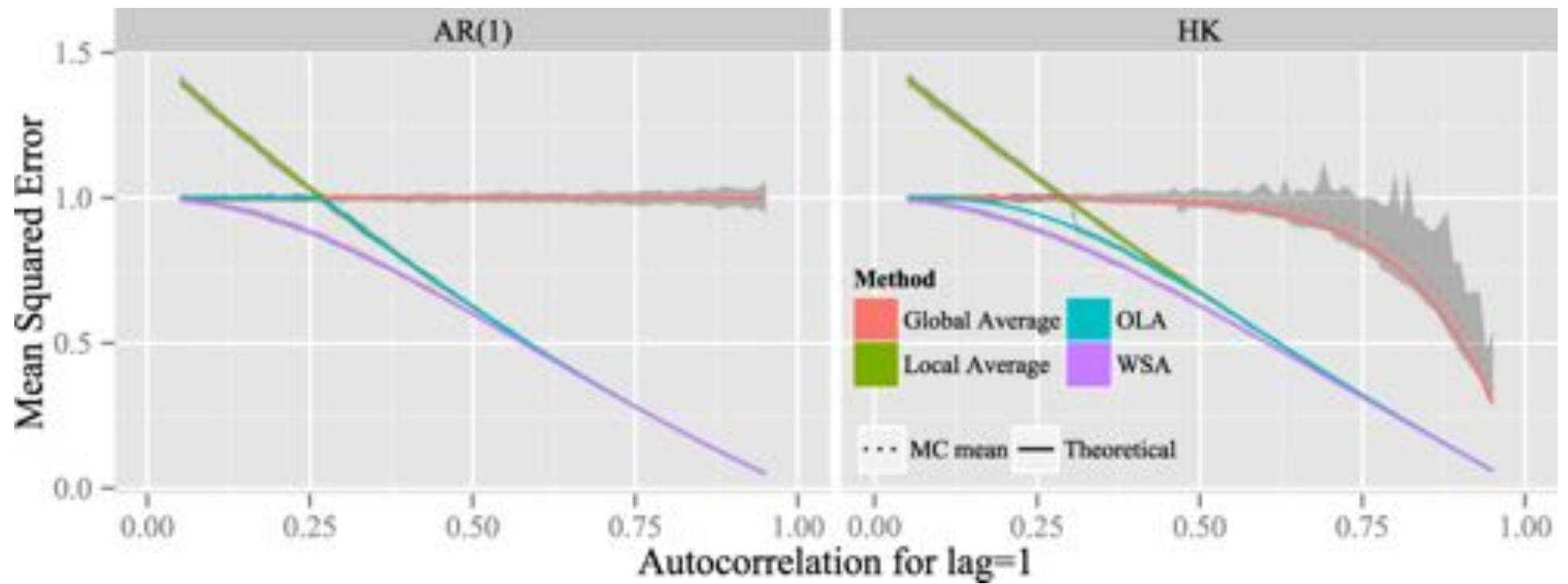


$$\underline{y} = \lambda_{HK} \frac{\sum_{i=-N}^N x_i}{2N} + (1 - \lambda_{HK}) \frac{x_{-1} + x_1}{2}$$



• For the case of **power-type ACS**, the λ - ρ_1 relationship **varies** significantly with time series length N .

• To circumvent this issue, the λ - ρ_1 is approximated using two additional parameters (λ_1, γ).



- We provide a **definitive argument against** the effortless use of **global (sample) mean** for infilling hydrometeorological (i.e., correlated) data.
- Local average ($n=1$) is preferable for:
 - **Markovian** processes with $\rho > 0.28$
 - **HK** processes with $\rho > 0.51$

Tobler's first law in geography:

“Everything is related to everything else, but near things are more related than distant things” [Tobler, 1970]

- A generalized framework, described by the **Weighted Sum of local and total Average** (WSA), is developed and its advantages are demonstrated.
- The **WSA** methodology is therefore tailored for a **quick** infilling of **sporadic gaps** in hydrometeorological time series.

- Dialynas, Y., P. Kossieris, K. Kyriakidis, A. Lykou, Y. Markonis, C. Pappas, S.M. Papalexiou, and D. Koutsoyiannis, Optimal infilling of missing values in hydrometeorological time series, *European Geosciences Union General Assembly 2010, Geophysical Research Abstracts, Vol. 12*, Vienna, EGU2010-9702, European Geosciences Union, 2010.
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Optimal infilling of missing values in hydrometeorological time series



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Thank you!

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