Physical basis of long range persistence in the climate system

Oscar J. Mesa and Vijay K. Gupta

Universidad Nacional de Colombia, Medellín, Colombia,
Area Metropolitana del Valle de Aburra, Medellín, Colombia,
CIRES, University of Colorado, Boulder, Colorado
ojmesa@unal.edu.co

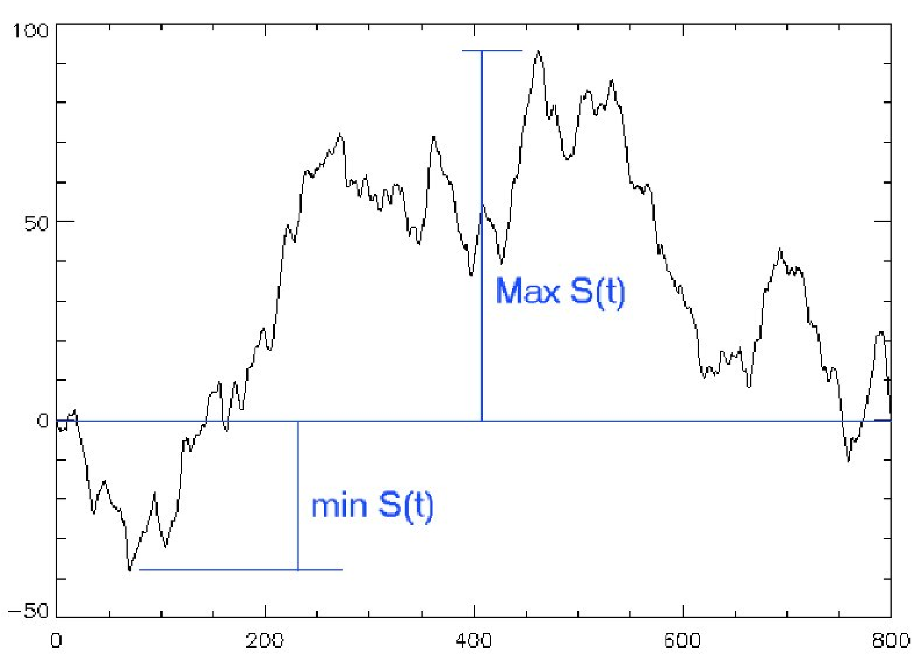
Abstract

The Hurst effect remains unsolved, despite its practical and theoretical importance.

Non uniformly mixing dynamical systems produce long range correlations when orbits expend long time near neutral fixed points in overall mixing systems.

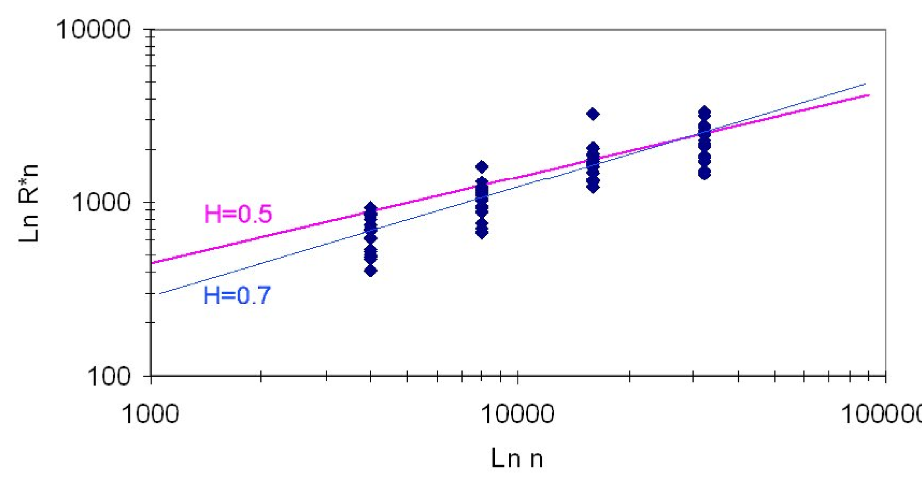
The simple auto oscillatory Daisyworld model that has been used to explore the importance of the atmosphere—biosphere feedbacks is one example. Depending on the parameters, time series resulting from the numerical integration of these examples exhibit the Hurst effect. We interpret this behavior as a consequence of the existence of fixed neutral points.

The Hurst Effect



Let q_t represent the average water inflow discharge over the period $(t - \Delta t,\ t)$, into an ideal reservoir, which is neither empty nor full. Let the record length be n, $\Delta t = 1$ and suppose that in each time period the same amount of water is extracted from the reservoir that is equal to the sample mean $\bar{q}$. Therefore, the continuity equation gives the volume of water stored in the reservoir at time t as, $S_t = S_{t-1} + (q_t - \bar{q})$. Let us further assume $S_0 = 0$. For these conditions, define the range of S_n as,

$$R_n = \max_{0 \le t \le n} S_t - \min_{0 \le t \le n} S_t. \qquad (1)$$



Harold E. Hurst was interested in estimating the design capacity of a reservoir for the Nile River in Egypt. He considered R_n as a measure of the reservoir capacity. He re-scaled the range by the sample standard deviation and called $R_n^* = R_n/\sigma$ the rescaled adjusted range. He analyzed records of different geophysical variables, and found that R_n^* grows with the record length n to a power h greater that 0.5, typically close to 0.7. Whereas according to the Functional Central Limit Theorem of Probability Theory for a very general class of stochastic processes, the following holds [Bhattacharya et al., 1983]

$$E[R_n^*] \sim (\frac{1}{2}\pi\theta n)^{0.5}, \text{ with } \theta = \sum_{k=-\infty}^{\infty} \rho_k. \qquad (2)$$

ρ_k denotes the sequence of lag–k correlation coefficients.

θ is also known as the "scale of fluctuation of the process" (Taylor [1922]).

The "Hurst Effect" is the deviation of the empirically observed values of power h from 0.5.

Because of the FCLT either non stationarity or long memory are necessary for the Hurst effect.

Finite θ, also called finite memory, depends on the rate of convergence of ρ_k to zero.

Examples

Fractional Brownian noise

A stationary Gaussian process with correlation function (Mandelbrot and Wallis [1969])
$\rho(s) = \frac{1}{2}\left(|s+1|^{2H} - 2|s|^{2H} + |s-1|^{2H}\right)$, with $H \in (0, 1)$. This process is the derivative of a fractional Brownian motion (FBM) $B_H(x)$, a non-stationary Gaussian process with normally distributed independent increments, $B_H(x) - B_H(y)$, with zero mean and variance $|x-y|^{2H}$.
The FBM is self–similar in the sense that $(B_H(x) - B_H(0)) \doteq s^{-H}(B_H(sx) - B_H(0))$.
Now, the local properties of realizations depend on $\rho(s)$ near $s = 0$. If $1 - \rho(s) \sim |s|^{-\alpha}$ as $s \to 0$, for some $\alpha \in (0, 2]$, then the fractal dimension of the trajectories in an m dimensional space is $D = m + 1 - \alpha/2$ [Gneiting and Schlather, 2004].
The asymptotic behavior of $\rho(s)$ at $s \to \infty$ determines presence or absence of long range dependence. If

$$\rho(s) \sim |s|^{-\beta} \text{ as } s \to \infty, \qquad (3)$$

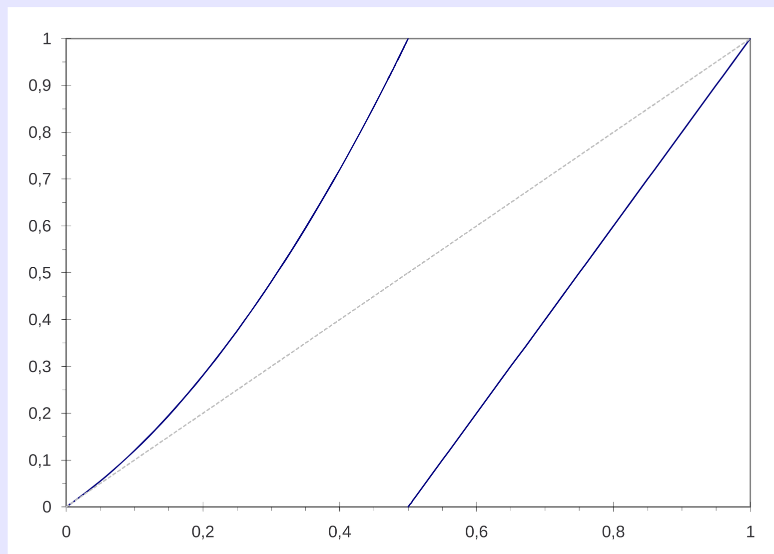
for some $\beta \in (0, 1]$, then $H = 1 - \beta/2$.

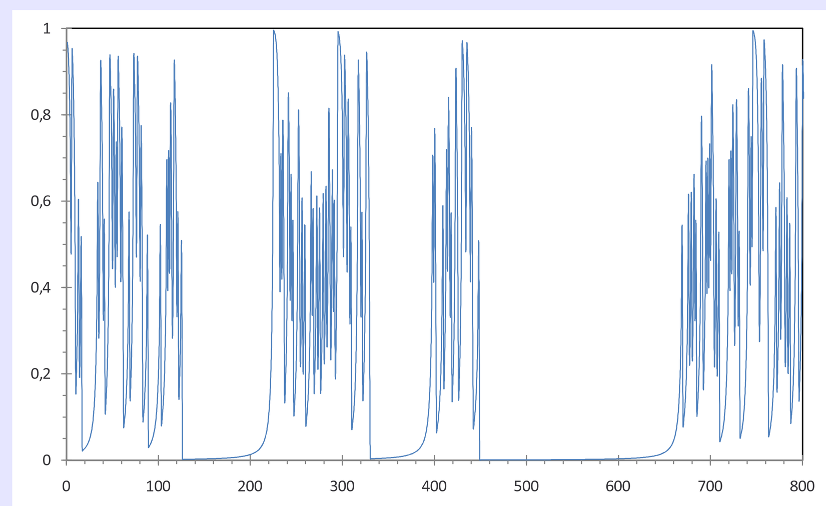
Non–stationary processes

Bhattacharya et al. [1983] proved that under certain class of non–stationarity the Hurst effect also arises in models with finite memory, or short-range dependence.

Intermittency maps

One dimensional interval maps, expanding at every point but at a neutral fixed point p, this means that $f(p) = p$ and that $|f'(x)| > 1$ for every $x \neq p$ and $f'(p) = 1$. The fixed point p is repelling, but nearby points remain close to p much longer than in the uniformly expanding case. This is characteristic of intermittency ([Alves and Viana, 2000; Luzzatto, 2004, 2003]). Intermittency maps exhibit slow rates of decay of correlations. Therefore this is a very simple deterministic dynamical system that produces the Hurst effect.





Long memory in Daisyworld

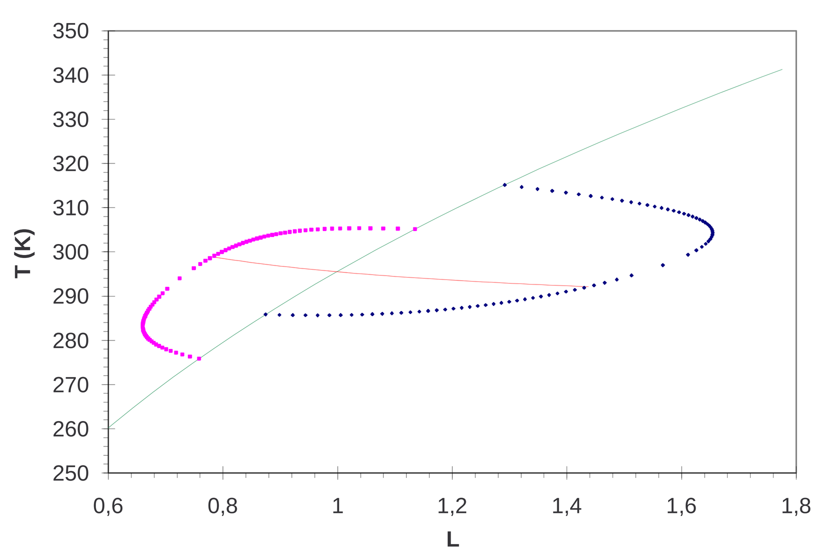
The Daisyworld is a simple but well-known physical model of climate [Watson and Lovelock, 1983; Lovelock, 1995; Wood et al., 2008]. We force the system with a time dependant deterministic quasi-periodic solar forcing that includes the annual cycle, and the effect of long-time changes in the orbital parameters of earth [Hartmann, 1994; Laskar et al., 2004]. Our experiments indicate that the forcing is necessary for the mixing to occur in the phase space, and for the trajectories to get near the neutral fixed points that produces long range correlations. In addition, we explore a case in which we add random noise to the solar forcing. It represents the effect of all those physical processes that are not explicitly represented in the dynamical equations.

H51B-0795, American Geophysical Union Fall Meting, December 2008, San Francisco, California

The differential equations and parameters taken from Nevison et al. [1999] are,

$$\frac{dT_e}{dt} = \frac{SL}{c_p}(1-A) - \frac{\varsigma}{c_p}T_e^4 \quad (4)$$
$$\frac{da_b}{dt} = a_b(x\beta_b - \gamma) \quad (5)$$
$$\frac{da_w}{dt} = a_w(x\beta_w - \gamma). \quad (6)$$

The dependent variables are the effective temperature $T_e\,[\mathrm{K}]$, the fraction of the planet cover by black and white daisies respectively a_b and a_w. The independent variable is time $t\,[\mathrm{yr}]$. Auxiliary variables that are used to facilitate in writing the differential equations include the planetary albedo, $A = xA_g + (1-p)A_s + a_bA_b + a_wA_w$; the fraction of fertile area that is not covered by daisies, $x = p - a_b - a_w$; and the growth rate of each species ($y = b$ or $y = w$) of daisy, $\beta_y = 1 - 0.003265(T_{op} - T_y)^2$, that depends on the local temperature of each species $T_y = q(A - A_y) + T_e$. The parameters are the solar constant S; the solar luminosity, $L = 1$ (present time); planetary heat capacity $c_p = 951\,\mathrm{Wm^{-2}K^{-1}yr}$; Stefan-Boltzmann constant $\varsigma = 5.67\times10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$; the death rate for both species of daisies, $\gamma = 0.3\,\mathrm{yr}^{-1}$; the albedos of fertile bare ground, $A_g = 0.5$, of non fertile areas, $A_s = 0.5$, of areas covered by black daisies, $A_b = 0.25$; and of areas covered by white daisies, $A_w = 0.75$; the fraction of potentially fertile area, $p = 1$; the optimal temperature for the daisies, $T_{op} = 295.5\,\mathrm{K}$; and the heat conduction coefficient among the different areas, $q = 20\,\mathrm{K}$. Numerical experiments were run using a Runge-Kutta integration scheme, with $\Delta t = 0.25$ yr, and going $125\,000$ years into the future.



There is a Hopf bifurcation at the point $L = L_2$ in the parameter space. At this point the eigenvalue is purely imaginary. As L grows, the system makes a continuous transition from a stable fixed point to a stable periodic orbit.

In next figure we present numerical results for two cases with different parameters L_0, near the bifurcation point $L = 0.7813$. In the upper panel a projection of the attractor is depicted, the next panel going from top to bottom presents the Hurst analysis using the so-called pox diagram, and the next panel illustrates the time series. It is clear from both the attractor and the time series that the trajectories spend a long time near the neutral fixed point. We wish to point to three important features: (i) the correlations exhibit a power law for the two values of L_0, (ii) for the value of $L_0 = 0.802$, the Hurst exponent is below 0.5, which is the unique value corresponding to an exponential decay or a zero correlation, (iii) $L_0 = 0.780$, the Hurst exponent is $0.67 > 0.5$, which corresponds to the case that Hurst observed in paleo hydro-climatic time series. Our results suggest the hypothesis that that the long memory in the observed time series is due to a neutral point in the physical dynamical system that is otherwise mixing.

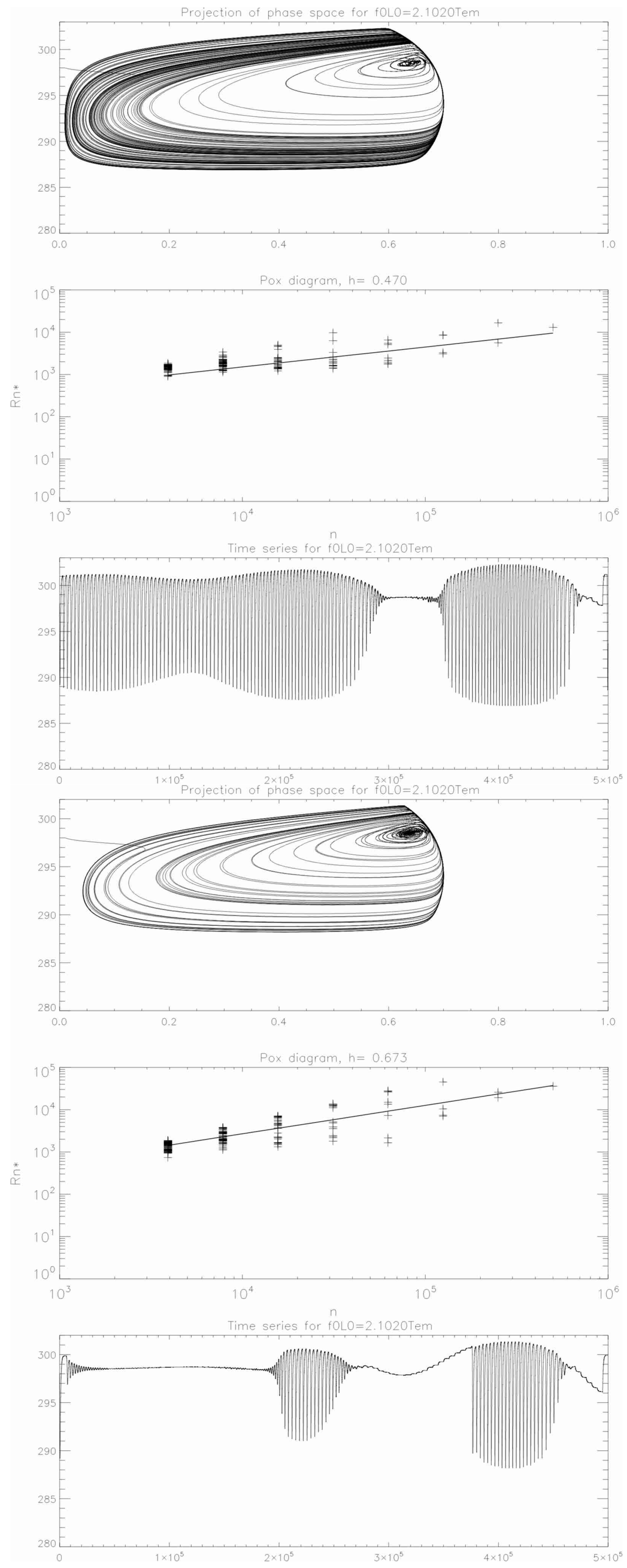


Figure 1: *Plane projection of the phase space for the forced Daisy world model (top), corresponding pox diagram (next from top to bottom) and time series(next) for two different values of the amplitude of the forcing: $L_0 = 0.802$ (top three panels), and $L_0 = 0.789$ (next three panels). The estimated Hurst coefficients are respectively 0.47 and 0.67. No random forcing*

We repeated the above analysis after adding random noise to the solar forcing but considered the case $L_0 = 0.802$, which is away from the bifurcation point. We also considered two values for the standard deviation of the noise, 2 and 10. The results are shown in Figure (2). For small values of the standard deviation the results are similar to the previous figure, which is to be expected. However, for the value of standard deviation equal to 10, the Hurst exponent increases from 0.47 to 0.76, which is greater than 0.5.

Theorem

Uniformly expanding maps have exponential decay of correlations

The mathematical theory of dynamical systems has a huge and growing literature, and we refer to Viana [1997] and Bonatti et al. [2005]for expository reviews.

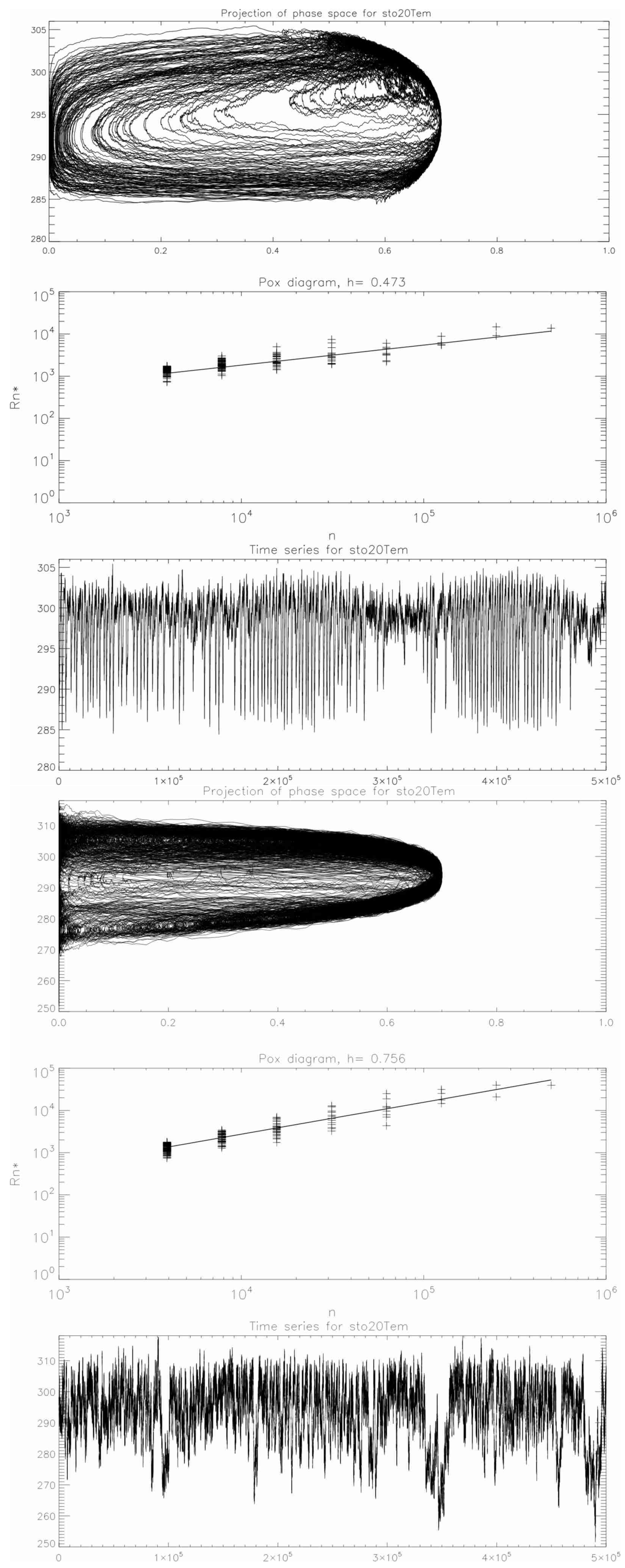


Figure 2: *Same as Figure 1 with random forcing with standard deviation, $\sigma = 2$ and 10 for the top three and bottom three panels respectively. The amplitude of the forcing is $L_0 = 0.802$ for both cases. The estimated Hurst coefficients are respectively 0.47 and 0.76*

References

J. F. Alves and M. Viana. Statistical Stability for Robust Classes of Maps With Non-Uniform Expansion, 2000. URL arXiv:math.DS/0011183.

R. N. Bhattacharya, V. K. Gupta, and E. Waymire. The hurst effect under trends. *Jour. Appl. Probability*, 1983.

C. Bonatti, L. Díaz, and M. Viana. Dynamics beyond uniform hyperbolicity. In J. Fröhlich, S. Novikov, and D. Ruelle, editors, *Encyclopaedia of Mathematical Sciences*, volume 102-III. Springer-Verlag, Berlin Heidelberg, 2005.

T. Gneiting and M. Schlather. Stochastic models that separate fractal dimension and the hurst effect. *SIAM REVIEW, Society for Industrial and Applied Mathematics*, 46(2):269–282, 2004.

D. L. Hartmann. *Global Physical Climatology*, volume 56 of *International Geophysics Series.* Academic Press, 1994.

J. Laskar, P. Robutel, F. Joutel, M. Gastineau, A. Correia, and B. Levrard. A long term numerical solution for the insolation quantities of the earth. *Astronomy and Astrophysics*, La-2004, 2004.

J. Lovelock. Geophysiology, the science of gaia. *Rev. of Geophys.*, 27: 215–222, 1995.

S. Luzzatto. Mixing and Decay of Correlations in Non-Uniformly Expanding Maps: a Survey of Recent Results, 2003. URL arXiv:math.DS/0301319.

S. Luzzatto. Stochastic-like Behaviour in Nonuniformly Expanding Maps, 2004. URL arXiv:math.DS/0409085.

B. Mandelbrot and J. Wallis. Some long run properties of geophysical records. *Water Res. Res.*, 5(2):321–340, 1969.

C. Nevison, V. Gupta, and L. Klinger. Self-susteained temperature oscillations on daisyworld. *Tellus*, 51B:806–814, 1999.

G. I. Taylor. Diffusion by continuous movements. *Proc. London Math. Soc.*, S2–20:196–212, 1922.

M. Viana. *Stochastic Dynamics of Deterministic Systems.* Lecture notes 21, Braz. Math. Colloq., IMPA, Rio de Janeiro, 1997.

A. J. Watson and J. Lovelock. Biological homeostasis of the global environment: the parable of daisyworld. *Tellus*, 35B:284–289, 1983.

A. J. Wood, G. J. Ackland, J. G. Dyke, H. T. P. Williams, and T. M. Lenton. Daisyworld: A review. *Rev. Geophys.*, 46:RG1001, 2008. doi: 10.1029/2006RG000217.